

On the Total Forcing Number of a Graph

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Abstract

A total forcing set in a graph G is a forcing set (zero forcing set) in G which induces a subgraph without isolated vertices. Total forcing sets were introduced and first studied by Davila [11]. The total forcing number of G , denoted $F_t(G)$ is the minimum cardinality of a total forcing set in G . We study basic properties of $F_t(G)$, relate $F_t(G)$ to various domination parameters, and establish NP -completeness of the associated decision problem for $F_t(G)$. Our main contribution is to prove that if G is a connected graph of order $n \geq 3$ with maximum degree Δ , then $F_t(G) \leq (\frac{\Delta}{\Delta+1})n$, with equality if and only if G is a complete graph $K_{\Delta+1}$, or a star $K_{1,\Delta}$.

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1 Introduction

In recent years, dynamic colorings of the vertices in a graph has gained much attention. Indeed, forcing sets [1, 10, 14, 17, 18, 19, 25, 28], k -forcing sets [2, 8], connected forcing sets [6, 7, 13], and power dominating sets [22, 29], have seen a wide verity of application and interesting relationships to other well studied graph properties. These aforementioned sets all share the common property that they may be defined as graph colorings that change during discrete time intervals. Of these dynamic colorings, we highlight that the most prominent is that of *forcing* (*zero forcing*), and the associated graph invariant known as the *forcing number* (*zero forcing number*). This paper continues the study of forcing in graphs by way of restricting the structure of forcing sets as an induced subgraph.

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *forcing process* is defined as follows: Let $S \subseteq V$ be a set of initially “colored” vertices, all other vertices are said to be “non-colored”. A vertex contained in S is said to be S -colored, while a vertex not in S is said to be S -uncolored. At each time step, if a colored vertex has exactly one non-colored neighbor, then this colored vertex *forces* its non-colored neighbor to become colored. If v is such a colored vertex, we say that v is a *forcing vertex*. We say that S is a *forcing set*, if by iteratively applying the forcing process, all of V becomes colored. We call such a set S an S -forcing set. In addition, if S is an S -forcing set in G and v is a S -colored vertex that forces a new vertex to be colored, then v is an S -*forcing vertex*. The cardinality of a minimum forcing set in G is the *forcing number* of G , denoted $F(G)$. If S is a forcing set which also induces a connected subgraph, then S is a *connected forcing set*. The cardinality of a minimum connected forcing set in G is the *connected forcing number* of G , denoted $F_c(G)$.

For graphs in general, it is known that computation of both $F(G)$ and $F_c(G)$ lie in the class of NP -hard decision problems, see [10, 28] and [6], respectively. Moreover, $F(G)$ and $F_c(G)$ have been related to many well studied graph properties such as minimum rank, independence, and domination, see for example [1, 2, 13]. For more on forcing and connected forcing, we refer the reader to [2, 6, 7, 8, 11, 12, 13, 14, 17, 18].

In this paper we study a variant of forcing. Namely, if $S \subseteq V$ is a forcing set of G that induces a subgraph without isolated vertices, then S is a *total forcing set*, abbreviated *TF-set*, of G . The concept of a total forcing set was first introduced and studied by the Davila in [11]. The minimum cardinality of a TF-set in G is the *total forcing number* of G , denoted $F_t(G)$. Minimum cardinality TF-sets in G are called $F_t(G)$ -sets.

We proceed as follows. In the next section concepts used throughout the paper are introduced and known facts and results needed are recalled. In Section 3 we provide fundamental properties of total forcing sets in graphs. Relationships between total forcing and various domination parameters is explored in Section 4. In Section 5, we study the effect on the total forcing number when the vertex whose removal creates no isolates is deleted from a graph. The computational complexity of the total forcing number is discussed in Section 6 where it is shown that the decision problem associated with total forcing is also NP -complete. We close in Section 7 with an upper bound on the total forcing number of a graph with minimum degree at least two in terms of the order and maximum degree of the graph.

2 Definition and Known Results

For notation and graph terminology, we will typically follow [24]. Throughout this paper, all graphs will be considered undirected, simple and finite. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$ of order $n = |V(G)|$ and size $m = |E(G)|$. Two vertices v and w are *neighbors* in G if they are adjacent; that is, if $vw \in E(G)$. The *open neighborhood* of a vertex v in G is the set of neighbors of v ,

denoted $N_G(v)$, whereas the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V(G)$ is the set of all neighbors of vertices in S , denoted $N_G(S)$, whereas the *closed neighborhood* of S is $N_G[S] = N_G(S) \cup S$. The *degree* of a vertex v in G , is denoted $d_G(v) = |N_G(v)|$. The minimum and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $S \subseteq V(G)$, the *degree of v in S* , denoted $d_S(v)$, is the number of vertices in S adjacent to v ; that is, $d_S(v) = |N(v) \cap S|$. In particular, $d_G(v) = d_{V(G)}(v)$. If the graph G is clear from the context, we simply write V , E , n , m , $d(v)$, $N(v)$, $N(S)$, δ and Δ rather than $V(G)$, $E(G)$, $n(G)$, $m(G)$, $d_G(v)$, $N_G(v)$, $N_G(S)$, $\delta(G)$ and $\Delta(G)$, respectively.

The distance between two vertices $v, w \in V$ is the length of a shortest (v, w) -path in G , and is denoted by $d_G(v, w)$. If no (v, w) -path exists in G , then we define $d_G(v, w) = \infty$. The maximum distance among all pairs of vertices of G is the *diameter* of G , denoted by $\text{diam}(G)$. The length of a shortest cycle in a graph G (containing a cycle) is the *girth* of G , denoted by $g = g(G)$. For a set of vertices $S \subseteq V$, the subgraph induced by S is denoted by $G[S]$. The subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$. If $S = \{v\}$, we simply write $G - v$ rather than $G - S$. We will denote the path, cycle, and complete graph on n vertices by P_n , C_n , and K_n , respectively. A *leaf* of G is a vertex of degree 1 in G .

A *packing* in a graph is a set of vertices that are pairwise at distance at least 3 apart; that is, if P is a packing in a graph G , and u and v are distinct vertices of P , then $d_G(u, v) \geq 3$. We note that if P is a packing, then the closed neighborhoods, $N_G[v]$, of the vertices v in P are pairwise vertex disjoint. A *perfect packing* (also called a *perfect dominating set* in the literature) is a packing that dominates the graph; that is, if P is a perfect packing, then the closed neighborhoods, $N_G[v]$, of the vertices v in P partition $V(G)$.

We use the standard notation $[k] = \{1, \dots, k\}$.

Domination in Graphs. A set of vertices $S \subseteq V$ is a *dominating set*, if every vertex not in S has a neighbor in S . The minimum cardinality of a dominating set in G is the *domination number* of G , denoted by $\gamma(G)$. If $S \subseteq V$ has the property that every vertex in G has a neighbor in S , then S is a *total dominating set*. The minimum cardinality of a total dominating set in G is the *total domination number* of G , denoted by $\gamma_t(G)$. If $S \subseteq V$ is a dominating set with the additional property that S induces a connected subgraph, then S is a *connected dominating set*. The minimum cardinality of a connected dominating set in G is the *connected domination number* of G , denoted $\gamma_c(G)$. A set S of vertices in a graph G is a (distance) *2-dominating set* if every vertex not in S is within distance 2 from some vertex of S . The (distance) *2-domination number*, written $\gamma_2(G)$, is the minimum cardinality of a 2-dominating set in G .

Domination and its variants are heavily studied in graph theory and we refer the reader to the monographs [20, 21, 24] which detail and survey many results on the topic. A survey on distance domination in graphs can be found in [23].

Serving as a dynamic approach to domination, the power domination process is defined as follows: For a given set of vertices $S \subseteq V$, the sets $(\mathcal{P}_G^i(S))_{i \geq 0}$ of vertices *monitored* by S at the i -th step are defined recursively by,

- (a) $\mathcal{P}_G^0(S) = N_G[S]$, and
- (b) $\mathcal{P}_G^{i+1}(S) = \cup\{N_G[v] : v \in \mathcal{P}_G^i(S) : |N_G[v] \setminus \mathcal{P}_G^i(S)| \leq 1\}$.

If $\mathcal{P}_G^{i_0}(S) = \mathcal{P}_G^{i_0+1}(S)$, for some i_0 , then $\mathcal{P}_G^j(S) = \mathcal{P}_G^{i_0}(S)$, for all $j \geq i_0$. We define $\mathcal{P}_G^\infty(S) = \mathcal{P}_G^{i_0}(S)$. If $\mathcal{P}_G^\infty(S) = V$, then S is a *power dominating set* of G . The minimum cardinality of a power dominating set in G is the *power domination number* of G , denoted by $\gamma_P(G)$.

Since every dominating set is also a power dominating set, and since every total dominating set is a dominating set, and since every connected dominating set is a total dominating set (provided $\gamma_c(G) \geq 2$), we make note of the following chain of inequalities.

Observation 1 *If G is a connected graph of order $n \geq 3$ and $\gamma_c(G) \geq 2$, then*

$$\gamma_P(G) \leq \gamma(G) \leq \gamma_t(G) \leq \gamma_c(G).$$

Known Results on Forcing. Recalling that computation of forcing for a general graph is NP -hard [6, 10, 28], we remark that finding computationally efficient lower and upper bounds for $F(G)$ and $F_c(G)$ has been of particular interest, see for example [2, 6, 7, 13]. In relation to this paper, we highlight that when considering efficient bounds for $F(G)$ and $F_c(G)$, both the minimum degree and maximum degree play an important role.

In the introductory paper on forcing in graphs, which first appeared due to a workshop on the minimum rank of a graph (AIM-Group) [1], it was shown that the minimum degree bounds $F(G)$ from below. That is, for any graph G with minimum degree δ , $F(G) \geq \delta$. This minimum degree lower bound has recently been significantly improved when the graph in question has restrictions on its girth and minimum degree. In particular, if G is a graph has minimum degree $\delta \geq 2$, Genter, Penso, Rautenbach, and Souza [17] and Gentner and Rautenbach [18] proved $F(G) \geq \delta + (\delta - 2)(g - 3)$, whenever $g \leq 6$, and also whenever $g \leq 10$ by Davila and Henning [12]. Using the techniques presented in [12], Davila, Malinowski, and Stephen [15] recently resolved this inequality in the affirmative for graphs with arbitrary girth.

As shown in [2], Amos, Caro, Davila, and Pepper, provided the first known upper bound on $F(G)$ in terms of maximum degree and order. In particular, if G is an isolate-free graph of order n and maximum degree Δ , they proved $F(G) \leq (\frac{\Delta}{\Delta+1})n$, and $F(G) \leq \frac{(\Delta-2)n+2}{\Delta-1}$, whenever the added restrictions that G is connected and $\Delta \geq 2$ are imposed. Moreover, they also showed that the forcing number of a graph is related to the connected domination number of G by way of the inequality $F(G) \leq n - \gamma_c(G)$.

Improving on these aforementioned maximum degree and order upper bounds, Caro and Pepper [8] gave a greedy algorithm which resulted in $F(G) \leq \frac{(\Delta-2)n - (\Delta-\delta) + 2}{\Delta-1}$, whenever G is connected and $\Delta \geq 2$. We remark that an analogous upper bound for $F_c(G)$ does not yet exist. Indeed, Davila, Henning, Magnant, and Pepper [13] asked whether or not there exists a function of order and maximum degree which bounds $F_c(G)$ from above.

The main aim of this paper is to introduce and study the total forcing number of a graph. In particular, we establish fundamental properties of $F_t(G)$, establish relationships between $F_t(G)$ and domination, prove NP -completeness of the total forcing decision problem, and provide a greedy algorithm for total forcing in connected graphs.

3 Fundamental Properties of $F_t(G)$

In this section we provide fundamental properties of total forcing. Since total forcing is not defined on graphs with isolated vertices, we shall assume throughout that all graphs contain no isolates. Since any forcing set on a single vertex will induce a graph with an isolated vertex, we observe that all TF-sets must contain at least two vertices.

Observation 2 *If G is an isolate-free graph, then $F_t(G) \geq 2$.*

As observed in [13], the only graphs with $F(G) = F_c(G) = 1$ are paths. Moreover, they also observed $\gamma_P(G) \leq F(G)$. Indeed, if G is not a path, then $F(G) \geq 2$ and $F_c(G) \geq 2$. Next observe that all TF-sets are forcing sets, and all connected forcing sets are TF-sets (provided $F_c(G) \geq 2$). We combine these observations and present them in the following chain of inequalities.

Observation 3 *If G is a connected graph that is not a path, then*

$$\gamma_P(G) \leq F(G) \leq F_t(G) \leq F_c(G).$$

We remark that Observation 3 yields an analogous chain of inequalities on forcing to that of domination presented in Observation 1.

Let $S \subseteq V$ be a forcing set of an isolate-free graph G . For each vertex $v \in S$, color exactly one neighbor of v . Call this coloring W . This resulting set is a superset of S , and hence, is also a forcing set. Moreover, $G[W]$ contains no isolated vertices. Hence, W is a TF-set. This observation implies that the total forcing number is no more than twice the forcing number. For example, a leaf of every non-trivial path is a forcing set of the path, while any two adjacent vertices on the path form a TF-set of the path, implying that for $n \geq 2$, $F_t(P_n) = 2$, $F(P_n) = 1$, and therefore $F_t(P_n) = 2F(P_n)$. We state our result formally as follows.

Observation 4 *If G is an isolate-free graph, then $F_t(G) \leq 2F(G)$, and this bound is sharp.*

We next classify the total forcing number of simple classes of graphs. As shown in [13], $F(C_n) = F_c(C_n) = 2$ and $F(K_n) = F_c(K_n) = n - 1$, whenever $n \geq 3$. This together with Observation 3 imply $F_t(C_n) = 2$ and $F_t(K_n) = n - 1$ whenever $n \geq 3$. We state these results formally with the following observation.

Observation 5 *For $n \geq 3$, the following holds.*

- (a) $F_t(P_n) = 2$.
- (b) $F_t(C_n) = 2$.
- (c) $F_t(K_n) = n - 1$.

We next observe that if G is a connected graph of order $n \geq 3$, then coloring any set of $n - 1$ vertices while leaving an arbitrary minimum degree vertex non-colored will result in a colored subgraph without isolates. Moreover, any colored vertices under such a coloring will have at most one non-colored neighbor, and hence will be a TF-set. Note that this coloring is best possible for the complete graph K_n of order $n \geq 3$. Hence, we establish the following simple upper bound.

Observation 6 *If G is an isolate-free graph of order $n \geq 3$, then $F_t(G) \leq n - 1$, and this bound is sharp.*

In light of Observation 6, it is natural to ask what graphs on a given number of vertices possess largest total forcing numbers? Before answering this question, we recall a characterization of $F_c(G) = n - 1$ presented in [7] and [13].

Theorem 7 ([7, 13]) *If G is a connected graph of order $n \geq 3$, then $F_c(G) = n - 1$ with equality if and only if $G = K_n$ for $n \geq 3$, or $G = K_{1,n-1}$ with $n \geq 4$.*

As an immediate consequence of Observations 3 and 6, and Theorem 7 above, we have the following result.

Theorem 8 *If G is a connected graph of order $n \geq 3$, then $F_t(G) = n - 1$, with equality if and only if $G = K_n$ for $n \geq 3$, or $G = K_{1,n-1}$ with $n \geq 4$.*

We establish next a useful property of a total forcing set in graphs that contain a vertex with at least two leaf neighbors.

Lemma 9 *If G is an isolate-free graph, then every vertex with at least two leaf neighbors is contained in every TF-set, and all except possibly one leaf neighbor of such a vertex is contained in every total forcing set.*

Proof. Let G be an isolate-free graph, and let v be a vertex of G with at least two leaf neighbors. Let S be an arbitrary TF-set of G . If the vertex v is not in S , then S contains at least one leaf neighbor of v . Such a leaf neighbor of v would be an isolated vertex in $G[S]$, and so the set S would not be a TF-set of G , a contradiction. Therefore, $v \in S$. If at least two leaf neighbors of v are not in S , then the vertex v cannot be an S -forcing vertex, implying that no S -uncolored leaf neighbor of v becomes colored during the forcing process, a contradiction. \square

4 Total Forcing versus Domination Parameters

It is well known that both the forcing number and connected forcing number of graphs are related to various domination parameters. In this section, we explore the relationship between total forcing and various domination parameters. We begin with the following result which relates the total forcing number to the domination number.

Theorem 10 *If G is a graph with minimum degree at least 3 and maximum degree Δ , then*

$$F_t(G) \leq \gamma(G)\Delta,$$

with equality if and only if G has a perfect dominating set and every vertex in this set has degree Δ in G .

Proof. Let G be a graph with minimum degree at least 3 and maximum degree $\Delta = \Delta(G)$, and let S be a minimum dominating set of G , and so $|S| = \gamma(G)$. Next let $\overline{S} = V \setminus S$, and let $W \subseteq V$ be the set of colored vertices obtained by coloring all of S , and for each $v \in S$, coloring all but one neighbor of v in \overline{S} . Observe that every vertex in S has at most one W -uncolored neighbor in \overline{S} . Further, each vertex of S with exactly one W -uncolored neighbor in \overline{S} is a W -forcing vertex. Moreover, since every vertex in G is either in S , or has a neighbor in S , it follows that every vertex of G is either in W , or is forced by a vertex in W . Since G has the property that $\delta \geq 3$, it follows that W induces a graph without isolates. Hence, W is a TF-set of G . We remark that the number of W -colored vertices is the number of vertices in S together with at most $d_G(v) - 1$ neighbors of each vertex $v \in S$ that belong to \overline{S} . Thus,

$$\begin{aligned} F_t(G) &\leq |W| \\ &\leq |S| + \sum_{v \in S} (d_G(v) - 1) \\ &\leq |S| + |S|(\Delta - 1) \\ &= |S|\Delta \\ &= \gamma(G)\Delta. \end{aligned}$$

As shown above, $F_t(G) \leq \gamma(G)\Delta$. Suppose next that $F_t(G) = \gamma(G)\Delta$. Thus we must have equality throughout the above inequality chain. In particular, $d_G(v) = \Delta$ for every

vertex $v \in S$. Further, the neighbors of each vertex $v \in S$ belong to \overline{S} and we color all but one, namely $d_G(v) - 1 = \Delta - 1$, such neighbors. If two vertices, say u and v , in S have a common neighbor, x say, then we color $\Delta - 1$ neighbors of u including the vertex x , and we color $\Delta - 2$ neighbors of v excluding the vertex x and one other arbitrary neighbor of v . Coloring $\Delta - 1$ neighbors of all other vertices in S different from u and v produces a TF-set of G with cardinality strictly less than $|S| + |S|(\Delta - 1)$, noting that the forcing vertex u is played in the time step before the forcing vertex v is played. This produces a TF-set of size less than $\gamma(G)\Delta$, a contradiction. Therefore, no two vertices in S are adjacent, or have a common neighbor. Thus, S is a perfect dominating set, and every vertex in S has degree Δ in G . \square

We next relate the total forcing number to the total domination number.

Theorem 11 *If G is an isolate-free graph with maximum degree $\Delta \geq 2$, then*

$$F_t(G) \leq \gamma_t(G)(\Delta - 1),$$

with equality if and only if G has a perfect total dominating set and every vertex in this set has degree Δ in G .

Proof. Let G be an isolate-free graph with maximum degree $\Delta = \Delta(G) \geq 2$. Let $S \subseteq V$ be a minimum total dominating set of G , and so $|S| = \gamma_t(G)$. Let $\overline{S} = V \setminus S$. Next let W be a set of colored vertices obtained by coloring S , and for each $v \in S$, coloring all but one neighbor of v in \overline{S} . Observe that every vertex in S has at most one W -uncolored neighbor in \overline{S} . Further, each vertex of S with exactly one W -uncolored neighbor in \overline{S} is a W -forcing vertex. Moreover, since every vertex in G is either in S , or has a neighbor in S , it follows that every vertex of G is either in W , or is forced by a vertex in W . Since each vertex of S has at least one neighbor in S , and we have only colored vertices in \overline{S} adjacent with vertices in S , we observe that W is a TF-set. We remark that the number of W -colored vertices is the number of vertices in S together with at most $d_G(v) - 2$ neighbors of each vertex $v \in S$ that belong to \overline{S} . Thus,

$$\begin{aligned} F_t(G) &\leq |W| \\ &\leq |S| + \sum_{v \in S} (d_G(v) - 2) \\ &\leq |S| + |S|(\Delta - 2) \\ &= |S|(\Delta - 1) \\ &= \gamma_t(G)(\Delta - 1). \end{aligned}$$

As shown above, $F_t(G) \leq \gamma_t(G)(\Delta - 1)$. Next suppose that $F_t(G) = \gamma_t(G)(\Delta - 1)$. Thus, we must have equality throughout the above inequality chain. In particular, $d_G(v) = \Delta$ for every $v \in S$. Further, each vertex $v \in S$ has all, except for exactly one, of its neighbors in \overline{S} and we color all but one, namely $d_G(v) - 2 = \Delta - 2$, such neighbors.

If two vertices, say u and v , in S have a common neighbor, x say, in \overline{S} , then we color $\Delta - 2$ neighbors of u in \overline{S} including the vertex x , and we color $\Delta - 3$ neighbors of v in \overline{S} excluding the vertex x and one other arbitrary neighbor of v in \overline{S} . Coloring $\Delta - 2$ neighbors of all other vertices in S (different from u and v) that belong to \overline{S} produces a TF-set of G with cardinality strictly less than $|S| + |S|(\Delta - 2)$, noting that the forcing vertex u is played in the time step before the forcing vertex v is played. This produces a TF-set of cardinality less than $\gamma_t(G)(\Delta - 1)$, a contradiction. Therefore, no two vertices in S have a common neighbor. Thus, the set S is a perfect total dominating set and every vertex in S has degree Δ in G . \square

The following result shows that the sum of the total forcing number and the (distance) 2-domination number of a graph is at most its order.

Theorem 12 *If G is a graph of order n with minimum degree at least 2, then*

$$F_t(G) + \gamma_2(G) \leq n.$$

Proof. Let G be a graph of order n with $\delta(G) \geq 2$. Let S be a minimum TF-set, and so $|S| = F_t(G)$. We show that the set $V(G) \setminus S$ is a 2-dominating set of G . If this is not the case, then there is a vertex $v \in S$ at distance at least 3 from every vertex outside S . Thus, every vertex within distance 2 from v belongs to the set S . We now consider the set $S' = S \setminus \{v\}$. If $G[S']$ contains an isolated vertex w , then since $G[S]$ is isolate-free this would imply that v is the only neighbor of w that belongs to the set S . However, the minimum degree at least two condition implies that w has at least one neighbor outside S . Such a neighbor of w that belongs to $V(G) \setminus S$ is at distance 2 from v in G , a contradiction. Therefore, $G[S']$ is isolate-free. Further since S is a TF-set and since the vertex v is never played in the forcing process, the set S' is a forcing set. Hence, the set S' is a TF-set, contradicting the minimality of the set S . We deduce, therefore, that the set $V(G) \setminus S$ is a 2-dominating set of G . Thus, $\gamma_2(G) \leq |V(G) \setminus S| = n - |S| = n - F_t(G)$. \square

We next recall a lemma which relates forcing sets to power dominating sets.

Lemma 13 ([11, 13]) *Let G be a graph. Then, $S \subseteq V$ is a power dominating set if and only if $N[S]$ is a forcing set of G .*

Using Lemma 13, we next establish two relationships between the total forcing number and the power domination number.

Theorem 14 *If G is an isolate-free graph with maximum degree Δ , then*

$$F_t(G) \leq \gamma_P(G)(\Delta + 1).$$

Proof. Let G be an isolate-free graph with maximum degree $\Delta = \Delta(G)$. Let $S \subseteq V$ be a minimum power dominating set of G ; that is, $|S| = \gamma_P(G)$. By the definition of power domination, see property (1) in the definition, we observe that S power dominates (observes) its closed neighborhood. Moreover, since G contains no isolated vertices, we observe that $\mathcal{P}_G^0 = N_G[S]$ is a set of vertices which induces a graph without isolated vertices. By Lemma 13, it follows that $\mathcal{P}_G^0 = N_G[S]$ is a forcing set. In particular, if we color $W = \mathcal{P}_G^0 = N_G[S]$, we obtain a TF-set. We remark the number of W -colored vertices is precisely the number of vertices in $N[S]$, i.e., we color all the vertices of S and at most $d_G(v)$ neighbors of each $v \in S$ that belong to \overline{S} . Thus,

$$\begin{aligned}
F_t(G) &\leq |W| \\
&\leq |S| + \sum_{v \in S} d_G(v) \\
&\leq |S| + |S|\Delta \\
&= |S|(\Delta + 1) \\
&= \gamma_P(G)(\Delta + 1). \quad \square
\end{aligned}$$

The following fundamental result in domination is attributed to Ore [27], and is often referred to as Ore's Theorem.

Theorem 15 (Ore's Theorem [27]) *If G is an isolate-free graph of order n , then $\gamma(G) \leq \frac{n}{2}$.*

Making use of Ore's Theorem, we next present another relationship between the power domination number and the total forcing number.

Theorem 16 *If G is an isolate-free graph, then $F_t(G) \geq 2\gamma_P(G)$, and this bound is sharp.*

Proof. Let G be an isolate-free graph and let $S \subseteq V$ be a $F_t(G)$ -set. Since S is a TF-set, we observe that $G[S]$ is an isolate-free graph. Thus, by Theorem 15, there exists a set of vertices $W \subseteq S$ which dominates $G[S]$ and has cardinality at most $|S|/2$. Since W dominates $G[S]$, it follows that $S \subseteq N[W]$. By Lemma 13, W is a power dominating set of G , and so $\gamma_P(G) \leq |W| \leq \frac{1}{2}|S| = \frac{1}{2}F_t(G)$. Rearranging, we get our desired result. To see that this bound is sharp, consider the cycle C_n on n vertices. \square

5 Effects of Vertex Removal on Total Forcing

If v is a vertex of an isolate-free graph G whose removal creates no isolates, it is natural to consider the effect on the total forcing number when the vertex v is deleted from G .

In this section, we discuss how the total forcing number of such a graph G relates to the total forcing number of $G - v$. We answer this with the following result.

Theorem 17 *If G is an isolate-free graph with maximum degree Δ that contains a vertex v such that $G - v$ contains no isolates, then*

$$F_t(G) - 2 \leq F_t(G - v) \leq F_t(G) + \Delta,$$

and these bounds are tight.

Proof. Let v be a vertex in the isolate-free graph of maximum degree Δ such that $G - v$ contains no isolates. We note that $\Delta \geq 2$. Let $T \subseteq V(G) \setminus \{v\}$ be a minimum TF-set of $G - v$, and so $|T| = F_t(G - v)$. Further, let w be an arbitrary neighbor of v in G . The set $T \cup \{v, w\}$ is a TF-set of G , implying that $F_t(G) \leq |T \cup \{v, w\}| \leq |T| + 2 = F_t(G - v) + 2$. Rearranging, we obtain $F_t(G) - 2 \leq F_t(G - v)$. This establishes the desired lower bound on $F_t(G - v)$.

We next prove the upper bound. Let $S \subseteq V(G)$ be a minimum TF-set of G , and so $|S| = F_t(G)$. Since S is a TF-set of G , the graph $G[S]$ contains no isolated vertex. Suppose that $v \notin S$. Immediately before the vertex v is played in the forcing process, at most one neighbor of v is uncolored. If immediately before the vertex v is played, no neighbor of v is uncolored, then the set $S \setminus \{v\}$ is TF-set of $G - v$, implying that $F_t(G - v) \leq |S| - 1 = F_t(G) - 1$. If immediately before the vertex v is played, exactly one neighbor, say w , of v is uncolored, then let w' denote an arbitrary neighbor of w different from v . In this case, the set $(S \setminus \{v\}) \cup \{w, w'\}$ is a TF-set of $G - v$, implying that $F_t(G - v) \leq (|S| - 1) + 2 = F_t(G) + 1 \leq F_t(G) + \Delta - 1$.

Hence, we may assume that $v \in S$, for otherwise the desired upper bound follows. Let S_v denote the set of neighbors of v in S , and so $S_v = N(v) \cap S$. By assumption, each neighbor of v has degree at least 2 in G . For each vertex $u \in S_v$, let u' be a neighbor of u different from v . We now consider the set,

$$S'_v = \bigcup_{u \in S_v} \{u'\}.$$

If $N_G(v) = S_v$ or if the vertex v is not played in the forcing process, then since S is a TF-set of G , the set $(S \setminus \{v\}) \cup S'_v$ is a TF-set of $G - v$, implying that $F_t(G - v) \leq |S| + |S'_v| - 1 \leq |S| + \Delta - 1 = F_t(G) + \Delta - 1 < F_t(G) + \Delta$. If the vertex v is played in the forcing process, then let v_1 denote the uncolored neighbor of v that becomes colored when v is played and let v_2 be an arbitrary neighbor of v_1 different from v . In this case, we note that at least one neighbor of v , namely v_1 , does not belong to the set S , and so $|S'_v| \leq d_G(v) - 1 \leq \Delta - 1$. Further, the set $(S \setminus \{v\}) \cup S'_v \cup \{v_1, v_2\}$ is a TF-set of $G - v$, implying that $F_t(G - v) \leq (|S| - 1) + |S'_v| + 2 \leq |S| + \Delta = F_t(G) + \Delta$.

That the upper bound of Theorem 17 is tight may be seen as follows. For $k \geq 3$, if G is the graph obtained by subdividing each edge of the star $K_{1,k}$ exactly once, then

$F_t(G) = k$ and $\Delta(G) = k$. However, if v is the unique (central) vertex of degree k in G , then $G - v$ is the graph consisting of k disjoint copies of P_2 . Thus, $F_t(G - v) = 2k = F_t(G) + \Delta$. When $k = 3$, the graph G is illustrated in Figure 2(a).

That the lower bound of Theorem 17 is tight may be seen by taking, for example, G to be the graph illustrated in Figure 2(b) where the vertex v is an arbitrary leaf. In this example, $F_t(G) = 5$ and $F_t(G - v) = 3 = F_t(G) - 2$. We remark that there are connected graphs G of arbitrarily large order achieving the lower bound of Theorem 17. For example, taking $k \geq 2$ vertex disjoint copies G_1, G_2, \dots, G_k of the graph G illustrated in Figure 2(b), where w_i denotes the vertex of G_i named w in G for $i \in [k]$. Let H be the graph formed by the disjoint union of these k graphs by adding the edges $w_j w_{j+1}$ for $j \in [k - 1]$. We note that $F_t(H) = 5k$ and $F_t(H - x) = 5k - 2 = F_t(H) - 2$ for every leaf x in H . \square

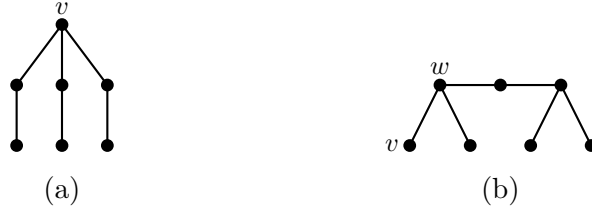


Figure 1: Graphs illustrating tightness of Theorem 17

We next show next that the upper bound of Theorem 17 can be improved if we remove a vertex of degree 1. More precisely, we show that removing a vertex of degree 1 from a connected graph of order at least 3 cannot increase the total forcing number.

Lemma 18 *If G is a connected graph of order at least 3 and v a vertex of degree 1 in G , then*

$$F_t(G - v) \leq F_t(G).$$

Proof. Let G is a connected graph of order at least 3. Let v be a vertex of degree 1 in G and let w be the neighbor of v . Since G has order at least 3, we note that w has at least two neighbors and that $G - v$ is a non-trivial graph. Let S be a minimum TF-set of G , and so $|S| = F_t(G)$. If $v \notin S$, then the set S is also a TF-set of $G - v$, and so $F_t(G - v) \leq |S| = F_t(G)$. Hence, we may assume that $v \in S$, for otherwise the desired result follows. Since $G[S]$ contains no isolated vertex, $w \in S$. If every neighbor of w belongs to S , then $S \setminus \{v\}$ is a TF-set of G , contradicting the minimality of S . Hence, there is a neighbor x of w that does not belong to S . Replacing v in S with the vertex x produces a new minimum TF-set, S' say, of G that is also a TF-set of $G - v$. Thus, $F_t(G - v) \leq |S'| = |S| = F_t(G)$. \square

6 Complexity of Total Forcing

As shown in the previous section, the total forcing number is related to a myriad of NP -complete graph invariants. In this section, we show that the decision problem associated with total forcing is also NP -complete. We state this decision problem formally as follows.

PROBLEM: Total forcing (TF)

INSTANCE: An isolate-free graph $G = (V, E)$ of order n and a positive integer $k \leq n$.

QUESTION: Does there exist a total forcing set $S \subseteq V$ of cardinality at most k ?

With the following theorem we prove that total forcing is NP -complete.

Theorem 19 *TF is NP-complete*

Proof. We first establish that TF is NP . Let G be an isolate-free graph of order n , and let $S \subseteq V$. Next observe that it can be checked in polynomial time if a vertex $v \in S$ has exactly one neighbor which is not in S ; that is, we may check if v is a S -forcing vertex in polynomial time. Next observe that there are at most $n - 1$ forcing steps during the forcing process in G . Hence, there exists a nondeterministic algorithm which may check in polynomial time whether S is a forcing set, whether S induces a graph without isolated vertices, and if S has cardinality at most k . It follows that TF is NP .

Next, we use for our reduction, the problem of forcing, which has been shown to be NP -complete in [10]. We state the decision problem of forcing below.

PROBLEM: Forcing (ZF)

INSTANCE: A simple graph $G = (V, E)$ of order n and a positive integer $k \leq n$.

QUESTION: Does there exist a forcing set of vertices $S \subseteq V$ with cardinality at most k ?

In order to prove that TF is NP -complete we construct a transformation from ZF to TF. Let $I = \langle G, k \rangle$ be an instance of ZF, where $G = (V, E)$ and $V = \{v_1, \dots, v_n\}$. Let G' be the graph obtained from G as follows. For each vertex v of G , add a vertex disjoint copy of a path P_3 and an edge from v to the (central) vertex of degree 2 in P_3 . For each vertex v_i of G , let G_i be the added copy of P_3 associated with v_i for $i \in [n]$. Further, let G_i be given the path $l_i^1 v_i^* l_i^2$, and so $v_i v_i^*$ is an edge of G' . The construction of the graph G' from the graph G is illustrated in Figure 2. We next define $f(I) = \langle G', k + 2n \rangle$.

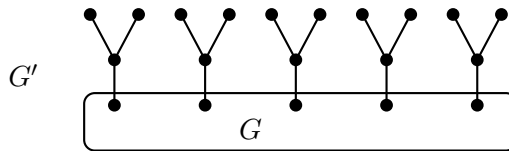


Figure 2: Obtaining G' from G

Next suppose that $I = \langle G, k \rangle$ is a “yes” instance of ZF . That is, G has a forcing set $S \subseteq V$ of cardinality at most k . We next show that $S' = S \cup \{v_1^*, \dots, v_n^*, l_1, \dots, l_n\}$ is a TF-set of G' . It is clear that G' is isolate-free since if v was isolated in G , it is now adjacent to the degree two vertex of some copy of P_3 . Moreover, each vertex S has a neighbor which is colored in some copy of P_3 , and hence, S' induces a graph with no isolated vertices. By Lemma 9, it is also clear that each v_i^* , and all but one of their respective degree one neighbors, at least l_i (say), are contained in every TF-set of G' , for $1 \leq i \leq n$; of which S' satisfies. Next observe that since each v_i^* is colored, any colored vertex in G , at any point during the forcing process starting in S , will have the same number of non-colored neighbors in G' . That is, we are assured that G as a subgraph will be completely vertex colored as a subgraph due to the forcing process starting at S' in G' . Moreover, once the vertices of G are colored in G' , each v_i^* , for $1 \leq i \leq n$, will have exactly one non-colored neighbor which is a leaf different than l_i , and hence will force. It follows that S' is a TF-set in G' . It follows that S' is a TF-set of G' with cardinality at most $k + 2n$, and hence, $f(I) = \langle G', k + 2n \rangle$ is a “yes” instance of TF.

Conversely, suppose $f(I) = \langle G', k + 2n \rangle$ is a “yes” instance of TF. That is, G' has a TF-set S' of cardinality $k + 2n$. By Lemma 9, we may assume $\{v_1^*, \dots, v_n^*, l_1, \dots, l_n\} \subseteq S'$. With this observation, we next observe for $1 \leq i \leq n$, it is the case that v_i^* is the unique vertex which forces its degree one neighbor different from l_i in G' . In particular, during the forcing process in G' any vertex which is colored in the subgraph G is colored by a vertex contained in G , i.e., the set $\{v_1^*, \dots, v_n^*, l_1, \dots, l_n\} \subseteq S'$ forces no vertex of G . It follows that $S = S' \setminus \{v_1^*, \dots, v_n^*, l_1, \dots, l_n\}$ is a forcing set of G . Since S' has cardinality at most $k + 2n$, it follows that S has cardinality at most k . In particular, we have that I is a “yes” instance of ZF. \square

7 A General Upper Bound

In this section we provide a sharp upper bound on the total forcing number of a graph in terms of its maximum degree and order. Recall that for a subset $S \subseteq V(G)$, the degree of v in S , denoted $d_S(v)$, is the number of vertices in S adjacent to v . We are now in a position to prove the following upper bound on the total forcing number of a graph.

Theorem 20 *If G is a connected graph of order $n \geq 3$ with maximum degree Δ , then*

$$F_t(G) \leq \left(\frac{\Delta}{\Delta + 1} \right) n,$$

with equality if and only if $G \cong K_{\Delta+1}$ or $G \cong K_{1,\Delta}$.

Proof. Let G be a connected graph of order $n \geq 3$ with maximum degree Δ . If $\Delta = 2$, then $G \cong P_n$ or $G \cong C_n$. In both cases, $F_t(G) = 2 \leq \frac{2}{3}n = (\frac{\Delta}{\Delta+1})n$, as desired.

Further, if $F_t(G) = (\frac{\Delta}{\Delta+1})n$, then we must have equality throughout this inequality chain, implying that $n = 3$ and $G \cong C_3 = K_3$ or $G \cong P_3$. Hence, we may assume that $\Delta \geq 3$. Among all maximum packings in G , let $P = \{v_1, \dots, v_k\}$ be chosen so that

- (1) the number of vertices not dominated by P is a maximum,
- (2) subject to (1), the sum of degrees of vertices in P is a maximum.

We note that if P is a perfect packing, then every vertex of G is dominated by P . Let A be the set of all neighbors of vertices in P ; that is, $A = N(P)$. For $i \in [k]$, let $A_i = N(v_i)$, and so $A = \cup_{i=1}^k A_i$. Further, let $|A_i| = \alpha_i$, and so $1 \leq \alpha_i = d_G(v_i) \leq \Delta$. Since P is a packing, we note that (A_1, \dots, A_k) is a partition of A . Further, $(A_1 \cup \{v_1\}, \dots, A_k \cup \{v_k\})$ is a partition of $N[P]$.

Let $B = V(G) \setminus N[P]$. If a vertex $w \in B$ is at distance at least 3 from every vertex of P , then the set $P \cup \{w\}$ would be a packing in G , contradicting the maximality of P . Hence, every vertex in B has a neighbor in A and is therefore adjacent to a vertex in A_i for some $i \in [k]$. Equivalently, the set A dominates the set B .

We now define a weak partition (B_1, \dots, B_k) of the set B as follows, where by a weak partition we mean a partition where some of the sets may be empty. Let B_1 be the set of all vertices in B that have a neighbor in A_1 . Thus, B_1 consists of all vertices $v \in B$ such that $d_{A_1}(v) \geq 1$. For $i \in [k] \setminus \{1\}$, let B_i be the set of all vertices in B that have a neighbor in A_i but no neighbor in A_j for any $j \in [i-1]$. Thus for $i \in [k] \setminus \{1\}$, the set B_i consists of all vertices $v \in B$ such that $d_{A_j}(v) = 0$ for all j where $1 \leq j < i$ and $d_{A_i}(v) \geq 1$. By definition, (B_1, \dots, B_k) is a weak partition of the set B .

If $d_G(v_i) = 1$ and $B_i = \emptyset$ for all $i \in [k]$, then $V(G) = P \cup A$, $|A| = |P| = k$ and $n = 2k$. In this case, by the connectivity of G and since each vertex of A has exactly one neighbor outside A , the set A is a TF-set of G , implying that

$$F_t(G) \leq |A| = k = \frac{1}{2}n < \frac{3}{4}n \leq \left(\frac{\Delta}{\Delta+1}\right)n.$$

Hence, we may assume that for at least one $i \in [k]$, $d_G(v_i) \geq 2$ or $d_G(v_i) = 1$ and $B_i \neq \emptyset$. Renaming the vertices in P if necessary, we may further assume that $d_G(v_i) \geq 2$ or $d_G(v_i) = 1$ and $B_i \neq \emptyset$ for all $i \in [r]$ where $1 \leq r \leq k$. Possibly, $r = k$. If $r < k$, then we note that $d_G(v_j) = 1$ and $B_j = \emptyset$ for all j where $r+1 \leq j \leq k$.

For $i \in [k]$, let G_i be the graph induced by $N[v_i] \cup B_i$, where recall that $N[v_i] = \{v_i\} \cup A_i$. Further, let G_i have order n_i . We first restrict our attention to those graphs G_i where $i \in [r]$. By definition of the sets A_i and B_i , we note that the set A_i dominates the set B_i . Let D_i be a minimum set of vertices in A_i that dominate B_i . We now consider two cases. In both cases, we note that necessarily $i \in [r]$.

Case 1. $B_i \neq \emptyset$. Let $|D_i| = d_i$, and note that $1 \leq d_i \leq \alpha_i$. Further, we note that if $D_i = A_i$, then $d_i = \alpha_i$, while if $D_i \subset A_i$, then $d_i < \alpha_i$. By the minimality of the set D_i , each vertex in D_i dominates a vertex in B_i that is not dominated by the other vertices in D_i . For each vertex $x \in D_i$, let x' be a vertex in B_i that is dominated in G_i by x but by no vertex of $D_i \setminus \{x\}$. Further, let

$$D'_i = \bigcup_{x \in D_i} \{x'\} \quad \text{and} \quad L_i = B_i \setminus D'_i.$$

Let $|L_i| = \ell_i$, and so $|B_i| = d_i + \ell_i$ and

$$n_i = |A_i| + |B_i| + 1 = \alpha_i + d_i + \ell_i + 1.$$

Each vertex in D_i is adjacent to v_i and to exactly one vertex in D'_i , and is therefore adjacent to at most $\Delta - 2$ vertices in L_i , implying that

$$\ell_i \leq |D_i|(\Delta - 2) = d_i(\Delta - 2).$$

We now consider two subcases.

Case 1.1. $D_i \subset A_i$. Let $|D_i| = d_i$. In this case, $1 \leq d_i < \alpha_i$. In particular, we note that $\alpha_i \geq 2$. Let w_i be an arbitrary neighbor of v_i that does not belong to D_i , and let $S_i = V(G_i) \setminus (D'_i \cup \{w_i\})$. By construction, the graph $G_i[S_i]$ is isolate-free. Further, the set S_i is a forcing set of G_i since if $x_1 = v_i$ and $D_i = \{x_2, \dots, x_{d_i+1}\}$, then the sequence $x_1, x_2, \dots, x_{d_i+1}$ of played vertices in the forcing process results in all vertices of G_i colored, where x_i denotes the forcing vertex played in the i th step of the process. In particular, we note that $x_1 = v_i$ is an S_i -forcing vertex that forces w_i to be colored, while each vertex in D_i is an S_i -forcing vertex that forces its unique neighbor in D'_i to be colored. Thus, the set S_i is a TF-set of G_i . Further, $|S_i| = \alpha_i + \ell_i$. As observed earlier, $0 < d_i < \alpha_i \leq \Delta$ and $\ell_i \leq d_i(\Delta - 2)$, implying that $\alpha_i + \ell_i \leq d_i\Delta + \alpha_i - 2d_i < d_i\Delta + \alpha_i \leq \Delta(d_i + 1)$. We note that the following holds.

$$\begin{aligned} |S_i| &< \left(\frac{\Delta}{\Delta+1}\right) n_i \\ \Updownarrow \\ \alpha_i + \ell_i &< \left(\frac{\Delta}{\Delta+1}\right) (\alpha_i + d_i + \ell_i + 1) \\ \Updownarrow \\ (\Delta + 1)(\alpha_i + \ell_i) &< \Delta(\alpha_i + d_i + \ell_i + 1) \\ \Updownarrow \\ \alpha_i + \ell_i &< \Delta(d_i + 1). \end{aligned}$$

Thus since $\alpha_i + \ell_i < \Delta(d_i + 1)$ holds, so too does the inequality $|S_i| < (\frac{\Delta}{\Delta+1})n_i$ hold. Thus, S_i is a TF-set of G_i satisfying the inequality

$$|S_i| < \left(\frac{\Delta}{\Delta+1}\right) n_i. \tag{1}$$

Case 1.2. $D_i = A_i$. In this case, $d_i = \alpha_i$, and so $n_i = 2\alpha_i + \ell_i + 1$ and $\ell_i \leq \alpha_i(\Delta - 2)$. Let $S_i = V(G_i) \setminus D'_i$. By construction, the graph $G_i[S_i]$ is isolate-free. Further, the set S_i is a forcing set of G_i since if $D_i = \{x_1, \dots, x_{\alpha_i}\}$, then the sequence x_1, \dots, x_{α_i} of played vertices in the forcing process results in all vertices of G_i colored, where x_i denotes the forcing vertex played in the i th step of the process. In particular, we note that each vertex in D_i is an S_i -forcing vertex that forces its unique neighbor in D'_i to be colored. Thus, the set S_i is a TF-set of G_i . Further, $|S_i| = \alpha_i + \ell_i + 1$. As observed earlier, $\alpha_i \geq 1$ and $\ell_i \leq \alpha_i(\Delta - 2)$, implying that $\alpha_i + \ell_i + 1 \leq \Delta\alpha_i - \alpha_i + 1 \leq \Delta\alpha_i$. We note that the following holds.

$$\begin{aligned}
|S_i| &\leq \left(\frac{\Delta}{\Delta+1}\right) n_i \\
\Updownarrow \\
\alpha_i + \ell_i + 1 &\leq \left(\frac{\Delta}{\Delta+1}\right) (2\alpha_i + \ell_i + 1) \\
\Updownarrow \\
(\Delta + 1)(\alpha_i + \ell_i + 1) &\leq \Delta(2\alpha_i + \ell_i + 1) \\
\Updownarrow \\
\alpha_i + \ell_i + 1 &\leq \Delta\alpha_i.
\end{aligned}$$

Thus since $\alpha_i + \ell_i + 1 \leq \Delta\alpha_i$ holds, so too does the inequality $|S_i| \leq \left(\frac{\Delta}{\Delta+1}\right)n_i$ hold. Thus, S_i is a TF-set of G_i satisfying the inequality

$$|S_i| \leq \left(\frac{\Delta}{\Delta+1}\right) n_i. \quad (2)$$

Case 2. $B_i = \emptyset$ and $\alpha_i \geq 2$. We note that in this case $i \in [r]$. We now let w_i be an arbitrary neighbor of v_i and let $S_i = N[v_i] \setminus \{w_i\}$. Since $\alpha_i \geq 2$, we note that $G_i[S_i]$ is isolate-free. Further since $B_i = \emptyset$, we have $n_i = \alpha_i + 1$. The resulting set S_i is a TF-set in G_i of size

$$|S_i| = \alpha_i = \left(\frac{\alpha_i}{\alpha_i + 1}\right) (\alpha_i + 1) \leq \left(\frac{\Delta}{\Delta + 1}\right) (\alpha_i + 1) = \left(\frac{\Delta}{\Delta + 1}\right) n_i,$$

noting that the vertex v_i is an S_i -forcing vertex that forces the vertex w_i to be colored in the first step of the forcing process. After the vertex v_i is played in the forcing process, all vertices in $V(G_i)$ become colored. Thus, S_i is a TF-set in G_i satisfying Inequality (2). This completes our discussion of Case 2.

We now return to the proof of Theorem 20. By the way in which the set S_i is constructed for each $i \in [r]$ (see Case 1 and Case 2 above), either $N[v_i] \subseteq S_i$ or $N[v_i] \setminus \{w_i\} \subseteq S_i$ for some neighbor w_i of v_i . We call such a vertex w_i an S_i -uncolored neighbor of v_i . Further, each such set S_i is a TF-set of G_i satisfying Inequality (1) or Inequality (2). We now let

$$S' = \bigcup_{i=1}^r S_i.$$

Further if $r < k$, we let

$$P' = \bigcup_{i=r+1}^k \{v_i\} \quad \text{and} \quad A' = \bigcup_{i=r+1}^k \{w_i\},$$

and so $A' = N(P')$ and $|A'| = |P'| = k - r$. Let A'' be the set of vertices in A' that are adjacent to no vertex of $A' \cup S'$. We note that if $w \in A''$, then since G is connected, the vertex w has a neighbor w' different from its (unique) neighbor in P' . Moreover, such a neighbor w' necessarily belongs to the set $A \setminus (A' \cup S')$. We note further that such a vertex w may have many neighbors w' different from its (unique) neighbor in P' . However, each such neighbor w' of w is an S_i -uncolored neighbor of v_i for some $i \in [r]$. Thus,

$$N(A'') \setminus P' \subseteq A \setminus (A' \cup S').$$

Let C be a minimum set of vertices in $A \setminus (A' \cup S')$ that dominates the set A'' . By the minimality of the set C , each vertex in C dominates a vertex in A'' that is not dominated by the other vertices in C . For each vertex $x \in C$, let x'' be a vertex in A'' that is dominated by x but by no vertex of $C \setminus \{x\}$. Further, let

$$A''' = \bigcup_{x \in C} \{x''\}.$$

We note that $A''' \subseteq A''$ and $|C| = |A'''|$. Further, the set C dominates A'' , and so each vertex of $A'' \setminus A'''$ has a neighbor in C . Let

$$S'' = C \cup (A' \setminus A''')$$

and consider the set

$$S = S' \cup S''.$$

We show that the set S is a forcing set in G . By construction, the graph $G[S]$ contains no isolated vertex. As shown earlier, each set S_i is a TF-set of G_i satisfying Inequality (1) or Inequality (2) for all $i \in [r]$. As observed earlier, no vertex of S_i is adjacent to any vertex of A'' for all $i \in [r]$. In particular, no vertex of S_i is adjacent to any vertex of A''' for all $i \in [r]$. By the way in which the set S_i is constructed for each $i \in [k]$, either $N[v_i] \subseteq S_i$ or $N[v_i] \cap S_i = N[v_i] \setminus \{w_i\}$ for some neighbor w_i of v_i . In the latter case, we call w_i the S -uncolored neighbor of v_i . Let $Q = \{v_1, \dots, v_r\}$, and note that if $r = k$, then $Q = P$, while if $r < k$, then $Q \subset P$.

In the first stage of the forcing process, we color all vertices in $N[Q]$ as follows. If v_1 has an uncolored S -neighbor (namely, w_1), then we play the vertex v_1 as our first vertex in the forcing process and consider next the vertex v_2 . Otherwise, if $N[v_1] \subseteq S_1$, then we immediately consider the vertex v_2 . If v_2 has an uncolored S -neighbor (namely, w_2), then we play the vertex v_2 in the forcing process and consider next the vertex v_3 . Otherwise, if $N[v_2] \subseteq S_2$, then we immediately consider the vertex v_3 . Proceeding in

this way, we first play the vertices in Q in order whenever such a vertex of P has an S -uncolored neighbor. In this way, all vertices in $N[Q]$ are colored.

In the second stage of the forcing process we color all S -uncolored vertices of B_1 . Thereafter, we color all S -uncolored vertices of B_2 , and so on, until we finally color all S -uncolored vertices of B_k . For this purpose for $i \in [r]$, let s_i be a sequence of played vertices, excluding the vertex v_i (which is already played in the first stage or is never played), in the forcing process in the graph G_i that results in all of $V(G_i)$ being colored, as defined in Case 1.1, Case 1.2 or Case 2. Let s be the sequence consisting of the sequence s_1 , followed by the sequence s_2 , followed by the sequence s_3 , and so on, finishing with the sequence s_r . Recall that after the first stage of the forcing process, all vertices in $N[Q]$ are colored. Playing the vertices of the sequence s_1 (in order) in the graph G forces all vertices of $V(G_1)$ to be colored in G . Thereafter, playing the vertices in the sequence s_2 (in order) forces all vertices of $V(G_2)$ to be colored in G . Continuing in this way, once the vertices in the sequence s_{i-1} have been played, we play the vertices in the sequence s_i (in order) thereby forcing all vertices of $V(G_i)$ to be colored in G for all $i \in [r] \setminus \{1\}$. We note that this is always possible since no vertex of S_i is adjacent to any vertex of A''' for all $i \in [r]$. This results in all vertices in $V(G_i)$ colored for all $i \in [r]$.

In the third stage of the forcing process, we play each of the vertices of C in turn, thereby coloring all vertices in A''' . Once the sequence of vertices in C are played, we play in the fourth and final stage of the forcing process each vertex in the set A' in turn, thereby coloring its (unique) neighbor in P' . In this way, starting with the initial set S , there is a sequence of vertices that we can play in the forcing process that results in all vertices of $V(G)$ colored. Thus, S is a TF-set of G .

It remains for us to verify that the set S satisfies $|S| \leq (\frac{\Delta}{\Delta+1})n$. Recall that $V(G_i) = \{v_i, w_i\}$ for all i where $r+1 \leq i \leq k$, and so $n_i = 2$ for such values of i . By our earlier observations, $|S_i| \leq (\frac{\Delta}{\Delta+1})n_i$ for $i \in [r]$, implying that

$$|S'| = \sum_{i=1}^r |S_i| \leq \sum_{i=1}^r \left(\frac{\Delta}{\Delta+1} \right) n_i.$$

Moreover, if $r < k$, then since $|C| = |A'''|$ and $|A'| = k - r$, we note that

$$\begin{aligned} |S''| &= |C| + |A'| - |A'''| \\ &= k - r \\ &= \sum_{i=r+1}^k 1 \\ &= \sum_{i=r+1}^k \frac{1}{2} n_i \\ &< \sum_{i=r+1}^k \left(\frac{\Delta}{\Delta+1} \right) n_i. \end{aligned}$$

Recall that $n = \sum_{i=1}^k n_i$. We note that either $r = k$, in which case

$$|S| = |S'| = \sum_{i=1}^k |S_i| \leq \sum_{i=1}^k \left(\frac{\Delta}{\Delta+1} \right) n_i = \left(\frac{\Delta}{\Delta+1} \right) n$$

or $r < k$, in which case

$$\begin{aligned} |S| &= |S'| + |S''| \\ &< \left(\sum_{i=1}^r \left(\frac{\Delta}{\Delta+1} \right) n_i \right) + \left(\sum_{i=r+1}^k \left(\frac{\Delta}{\Delta+1} \right) n_i \right) \\ &= \sum_{i=1}^k \left(\frac{\Delta}{\Delta+1} \right) n_i \\ &= \left(\frac{\Delta}{\Delta+1} \right) n. \end{aligned}$$

Thus, $F_t(G) \leq |S| \leq \left(\frac{\Delta}{\Delta+1} \right) n$, as desired. Suppose next that $F_t(G) = \left(\frac{\Delta}{\Delta+1} \right) n$. Thus, S is a minimum TF-set in G , and $|S| = \left(\frac{\Delta}{\Delta+1} \right) n$. Recall that by our earlier assumptions, $\Delta \geq 3$. If $r < k$, then as shown above $|S| < \left(\frac{\Delta}{\Delta+1} \right) n$, a contradiction. Hence, $r = k$, implying that $|S_i| = \left(\frac{\Delta}{\Delta+1} \right) n_i$ for all $i \in [k]$. This implies that the set S_i must have been constructed in Case 2 for all $i \in [k]$; that is, $B_i = \emptyset$ for all $i \in [k]$. Thus, $B = \cup_{i=1}^k B_i = \emptyset$, implying that the packing P is a perfect packing. By our choice of the packing P , this implies that every maximum packing in G is a perfect packing.

Suppose that v_i is a vertex in P of degree at least 2 for some $i \in [k]$. If the set S_i is constructed as in Case 1.2, then strict inequality holds in Inequality (2) since in this case $\alpha_i > 1$ implying that $\alpha_i + \ell_i + 1 < \Delta \alpha_i$. Hence, the set S_i must have been constructed as in Case 2. Further for equality to hold in Inequality (2), the vertex v_i has maximum possible degree, namely Δ ; that is, $d_G(v_i) = \alpha_i = \Delta$. By the way in which S_i is constructed (see Case 2), the set S_i contains all but one vertex from the sets $N[v_i]$, and so $|S_i| = n_i - 1 = \Delta$.

Suppose that $d_G(v_i) = 1$ and $B_i \neq \emptyset$ for some $i \in [k]$. Adopting our earlier notation, this implies that the (unique) neighbor w_i of v_i has degree Δ and every neighbor of w_i in G different from v_i belongs to B_i . Renaming the vertices in P if necessary, we may choose $i = k$. If $k = 1$, then we obtain a contradiction since the packing consisting of the singleton vertex w_1 of degree Δ would contradict our choice of P which currently consists of the singleton vertex v_1 of degree 1. Hence, $k \geq 2$.

Since G is connected, we can choose the neighbor w'_k of w_k so that it has at least one neighbor outside $V(G_k)$. Since $w'_k \in B_k$, we note that w'_k is not adjacent with a neighbor of v_j for any $j < k$ for otherwise $w'_k \in B_j$ for some $j \in [k-1]$, a contradiction.

However, if remove the vertex v_k from the set $S_k = V(G_k) \setminus \{w'_k\}$ and immediately before playing the vertex w_k in the forcing process we play a neighbor of w'_k outside $V(G_k)$ (which forces w'_k to be colored) and then play the vertex w_k (which forces v_k to be colored), we produce a new TF-set of G of cardinality $|S| - 1$, a contradiction.

Hence, $d_G(v_i) \geq 2$ for all $i \in [k]$. Suppose that $k \geq 2$. (Recall that $|P| = k$.) We note that $(N[v_1], N[v_2], \dots, N[v_k])$ is a partition of $V(G)$. As observed earlier, each vertex v_i has Δ neighbors for all $i \in [k]$; that is, $|N(v_i)| = \Delta$. Since G is connected, there is a neighbor, u_1 say, of v_1 that is adjacent to a vertex outside $N[v_1]$. Renaming indices if necessary, we may assume that u_1 and v_2 have a common neighbor, say w_2 . Since u_1 has degree at most Δ and is adjacent to v_1 , at least one neighbor of v_i , say u_i , is not adjacent to u_1 for all $i \in [k] \setminus \{1\}$.

We now consider the set $T = V(G) \setminus \{u_1, u_2, \dots, u_k, w_2\}$. Since $\Delta \geq 3$, the graph $G[T]$ is isolate-free noting that v_j and at least one of its neighbors belong to T for all $j \in [k]$. We show that T is a forcing set. The first vertex played in the forcing process is the vertex v_1 which colors u_1 . The second vertex played is the vertex u_1 which colors w_2 . As this stage of the forcing process all vertices are colored except for the vertices u_2, \dots, u_k . For $i \in [k] \setminus \{1\}$, we play as the $(i + 1)$ st move in the forcing process the vertex v_i which colors the vertex u_i . Once the vertices v_2, \dots, v_k have been played (after $k + 1$ steps in the forcing process) all vertices of $V(G)$ are colored. Hence, T is TF-set of G . However, $|T| = n - (k + 1) = |S| - 1$, and so $|T| < |S| = F_t(G)$, a contradiction. Therefore, $k = 1$.

Hence, $k = 1$ and the vertex v_1 has degree Δ . If $G \not\cong K_{\Delta+1}$ and $G \not\cong K_{1,\Delta}$, then v_1 has three neighbors x_1, y_1 and z_1 such that x_1 is adjacent to y_1 but not to z_1 . In this case, the set $S^* = V(G) \setminus \{y_1, z_1\}$ is a TF-set of G , noting that starting with the set S^* we play as our first vertex in the forcing process the vertex x_1 which forces the vertex y_1 to be colored, and as our second vertex the vertex v_1 which forces the vertex z_1 to be colored. Thus, $F_t(G) \leq |S^*| < |S| = F_t(G)$, a contradiction. Therefore, $G \cong K_{\Delta+1}$ or $G \cong K_{1,\Delta}$. This completes the proof of Theorem 20. \square

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